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On duality walls in string theory

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Abstract: Following the RG flow of an $\mathcal{N}=1$ quiver gauge theory and applying Seiberg duality whenever necessary defines a duality cascade, that in simple cases has been understood holographically. It has been argued that in certain cases, the dualities will pile up at a certain energy scale called the duality wall, accompanied by a dramatic rise in the number of degrees of freedom. In string theory, this phenomenon is expected to occur for branes at a generic threefold singularity, for which the associated quiver has lorentzian signature. We here study sequences of Seiberg dualities on branes at the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold singularity. We use the naive beta functions to define an (unphysical) scale along the cascade. We determine, as a function of initial conditions, the scale of the wall as well as the critical exponent governing the approach to it. The position of the wall is piecewise linear, while the exponent appears to be constant. We comment on the possible implications of these results for physical walls.

Keywords: Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories.

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1. Introduction

Supersymmetric gauge theories are the most promising candidates for grand unification in particle physics, and their dynamics is therefore an important area of research. An even more interesting subclass are gauge theories that can be embedded into string theory, i.e., theories describing the low energy dynamics of (decoupled subsectors of) appropriately compactified and branified superstring theories. This subclass is distinguished by the fact that string theory provides, in principle, the unification with gravity.

At the basis of unification of course lie the phenomenon and generalized notion of scale dependence of physical theories. Effective gauge couplings and other physical parameters depend on the energy scale at which they are measured, in a way that is determined by renormalization group (RG) flow. Moreover, at certain energy scales, even the elementary degrees of freedom can change, leading to confinement and chiral symmetry breaking. Prominent in supersymmetric gauge theories is the possibility of duality. In particular, Seiberg duality [1] is the statement that a collection of different $\mathcal{N}=1$ supersymmetric gauge theories with gauge group and matter content related in a particular way can provide different microscopic definitions of one and the same underlying theory. Which microscopic description is appropriate depends, again, on the energy scale.

We will here be concerned with one class of four-dimensional gauge theories that can be embedded in string theory, namely through D-branes at singularities of Calabi-Yau threefolds. For such theories, some of the above questions have been entirely reformulated in recent years in the light of the AdS/CFT correspondence. For example, RG flow and duality can be realized directly as the dependence of certain supergravity fields on the portion of the dual geometry one is considering. Seiberg duality, in particular, arises from the fact that a gauge theory interpretation is possible only if the periods of supergravity form fields lie in a certain range [3]. Using gauge symmetries to shift the periods corresponds on the D-brane side to a change of basis of fractional branes at the singularity, as recently explained in [8].

In this paper, we consider quiver gauge theories arising from D-branes at the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold singularity. This is an interesting example because it is among the simplest singularities that is intrinsically three-dimensional (i.e., not related to an ADE singularity on K3), and some of the important ingredients of $\mathcal{N}=1$ theories in four dimensions, such as chiral anomaly cancellation, appear there for the first time. Of importance to us here is the fact that the associated quiver is hyperbolic, i.e., with indefinite Cartan matrix, whereas for simple singularities the quiver is elliptic or at best parabolic. See [6] for a definition and discussion of these terms.

The goal underlying our study is to understand RG flows, their cascades, and their walls, for quivers of generic threefold singularities. Here, as in [3], we refer as a duality cascade to the sequence of Seiberg dualities that are necessary along the RG flow in order to keep a gauge theory interpretation at every energy scale. Such a cascade was studied in [3], where the theory was the affine A_1 quiver, dual to the conifold geometry. More such flows were analyzed from a purely gauge theory point of view by Fiol [6]. It is pointed out in [6] that for a generic hyperbolic quiver, the scales at which one has to perform Seiberg duality pile up at a certain finite energy scale called a duality wall [7]. The existence of this wall casts some doubt at the possible UV completion of the theory.

All quivers whose cascades have been analyzed so far in [3, 6] are non-chiral, and either are elliptic or parabolic, or else are hyperbolic but have no apparent embedding into string theory, which might make the existence of the wall a little less worrisome. Our example is probably the simplest that is at the same time chiral, hyperbolic, and can be embedded in string theory. The naive procedure that we will use illustrates the behavior of duality cascades, and in particular the appearance of walls, in such theories.

Our present results can, however, not be viewed as support for the existence of duality walls in string theory.[1] The only anomaly-free brane configuration at the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold is the regular D3-brane, and this theory is conformal. After Seiberg duality, the naive beta functions do not vanish anymore, and one might be led to the conclusion that one can induce a cascade in this way. But a more careful analysis involving the exact beta

[1]We thank Andreas Karch for a discussion on the role of anomalous dimensions.

function shows that the Seiberg dual theories are, in fact, also conformal and do not flow. In contrast, our prescription involves following the *unphysical* flow induced by the naive beta functions. This is the same procedure that was also used in [5] for the same quiver as ours, and in [6] for other theories, for which it is actually more justified. In [5], the possible set of Seiberg dualities was presented as a tree originating in the IR with an infinite number of branch points as one proceeds to the UV. In fact there is no RG flow along this tree. Using the exact beta functions it is easy to see that all theories on this tree are conformal. One can view the naive flows that we study here as an organizing principle for trees of Seiberg dualities.

The question we shall ask is a simple thermodynamical one. Along the flow, how does the number of degrees of freedom depend on the scale? Obviously, this question only makes sense before the wall, so we first have to know its position. As we shall see, the position of the wall depends on the initial conditions that we specify. For our example, this dependence is, surprisingly, simply piecewise linear, see figure 2 in section 5. On the other hand, the approach to the wall appears to be universal. The critical exponent measuring the number of degrees of freedom is one, independent of initial conditions.

As mentioned above, the appearance of walls is unphysical in our example. This is the simplest example of a del Pezzo quiver, as mentioned in [5]. It would be interesting to analyze the RG flow behavior of the higher del Pezzos. Since there is more freedom regarding anomaly cancellation, one can imagine that some configurations will actually exhibit physical duality walls. On the other hand, it might be true that walls are simply absent generically in the UV behavior of gauge theories that appear in string theory. We hope to return to this problem in the future.

We conclude this introduction with a few speculations concerning the physical relevance of duality walls. If duality walls turn out to exist in string theory, this raises a number of interesting questions. For example, in a holographic picture, counting the number of degrees of freedom is related to the entropy of black holes (branes). See [4] for an exposition of this philosophy in the context of the Klebanov-Strassler flow. If the number of degrees of freedom diverges at a certain energy scale, the corresponding black holes must be very peculiar. Moreover, the divergence of the number of degrees of freedom for an observer probing at the scale of the wall would imply the existence of highly mysterious singular points in the closed string moduli space. This might also indicate the emergence of new effective degrees of freedom as the UV completion of the theory.

Last but not least, we mention another, more mathematical, aspect of our work. It is by now well-appreciated that quivers, their algebras and their representations have an intrinsic connection to geometry in the context of D-branes. see e.g., [13, 14, 2, 9, 15, 16] The representation theory of quivers is a rather important but very hard, branch of mathematics, see e.g., [17]. Just to mention one aspect, the group of Seiberg duality transformations is the natural analog of the Weyl group whose role in the representation theory of Lie algebras is familiar. Understanding this group can be very difficult. It is a natural question to ask whether the duality cascades (subgroups of the duality group) induced by RG flow play any particular role in the general representation theory of quivers.

2. The underlying quiver

In this paper, we study the properties of the tree of Seiberg dualities of certain four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories embeddable in string theory through fractional branes at singularities. Specifically, our gauge theory will be a quiver theory, with quiver depicted in figure 1. The gauge group contains three factors $\mathrm{U}(n_i)$, $i = 1, 2, 3$, and there are f_i chiral multiplets in bifundamentals as shown.

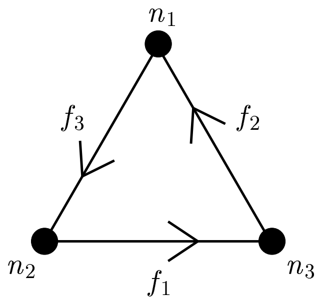

Figure 1: The quiver.

There is also a superpotential, which is very important for understanding the moduli space and the dynamics of the quiver. In particular, it determines the anomalous dimensions of various chiral fields. As mentioned in the introduction, we shall here use naive beta functions without taking into account the effect of the superpotential.

2.1 D-brane interpretation

The gauge theory that canonically describes N (regular) D3-branes on the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold is a quiver theory of the above type with $(n_1, n_2, n_3) = (N, N, N)$ and $(f_1, f_2, f_3) = (3, 3, 3)$ [2]. This has the D-brane interpretation that the D3-brane can be obtained as a bound state of the three fractional branes e_i associated with each of the nodes in figure 1, i.e., the e_i correspond to the gauge theories with $n_i = 1$ and $n_j = 0$ for $j \neq i$.

More generally, we can study arbitrary bound states (n_1, n_2, n_3) of the three fractional branes, with $f_i = 3$ held fixed. In fact, it is claimed that all (spacetime filling) branes on the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold can be constructed as bound states of the elementary branes (e_i), at least at zero string coupling. One can also discuss, along similar lines, the spectrum of D-branes at different points in the Kähler moduli space of the non-compact Calabi-Yau $\mathcal{O}_{\mathbb{P}^2}(-3)$ into which the orbifold can be blown up. We refer to [9] for a general investigation of the D-geometry of $\mathbb{C}_3/\mathbb{Z}_3$.

The point of interest here is that the $(n_1, n_2, n_3) = (N, N, N)$, $f_i = 3$ quiver is not the only possible theory whose moduli space yields $\mathbb{C}_3/\mathbb{Z}_3$. Indeed, there is an infinite number of possible duality transformations that one can apply to the above quiver which give entirely equivalent descriptions. The D-brane interpretation of this is that of a *change of basis of elementary branes* [8]. In other words, a given brane can be constructed either as bound state of (n_1, n_2, n_3) elementary branes e_i with f_i chiral multiplets, or as bound state of (n'_1, n'_2, n'_3) branes e'_i with f'_i chiral multiplets. Since the Ramond-Ramond charge of the given brane does not depend on the way we write it, we require that at the level of RR charges, $\sum n_i e_i = \sum n'_i e'_i$. In other words, if we have,

$$e'_i = \sum B_{ij} e_j \,, \tag{2.1}$$

then

$$n'_i = \sum n_j B^{-1}_{ji} \,. \tag{2.2}$$

To determine the number of chiral multiplets after the change of basis, we note that we can assemble the f_i into a 3×3 matrix I_{ij} which has the natural interpretation of intersection form of cycles in the resolution $\mathcal{O}_{\mathbb{P}^2}(-3)$ of the orbifold singularity $\mathbb{C}_3/\mathbb{Z}_3$, see e.g., [9]. As a consequence, we can read off the number of chiral multiplets f'_i from the intersection form in the new basis, i.e.,

$$I' = BIB^T . \tag{2.3}$$

In the case at hand, as in many other examples, it turns out that the total collection of quivers that give $\mathbb{C}_3/\mathbb{Z}_3$ can be characterized by the solutions of a certain Diophantine equation, which here is [13, 5, 11]

$$n_1^2 + n_2^2 + n_3^2 = 3n_1 n_2 n_3 . \tag{2.4}$$

Namely, if (n_1, n_2, n_3) is a solution of this equation with g.c.d.$(n_1, n_2, n_3) = 1$, then N D3-branes on $\mathbb{C}_3/\mathbb{Z}_3$ are given by the quiver with charge vector $N(n_1, n_2, n_3)$ and $f_i = 3n_i$. In this way, each solution of (2.4) gives rise to a basis of fractional branes on $\mathbb{C}_3/\mathbb{Z}_3$. Of course, it does not specify the superpotential, but as we noted before, we shall neglect it here.

2.2 Comments on RG flow

Not every possible representation of the quiver gives rise to a physical gauge theory at non-zero coupling. We also need to satisfy the anomaly cancellation condition for every node (number of incoming arrows equals number of outgoing arrows), i.e.,

$$n_{i+1} f_{i-1} = n_{i-1} f_{i+1} \tag{2.5}$$

for $i = 1, 2, 3$ (we take $i = 1, 2, 3 \bmod 3$), in other words, we must have $(f_1, f_2, f_3) \propto (n_1, n_2, n_3)$. Thus, there is only one configuration for which anomalies are cancelled, which are exactly the D3-branes. We shall therefore restrict ourselves to these configurations.

The gauge theory on these D3-branes is a conformal theory. For the canonical description, $(n_1, n_2, n_3) = (N, N, N)$, this follows immediately from the naive beta functions, which read in general

$$\frac{d(1/g_i^2)}{d \ln \mu} = \beta_i = 3N_{c,i} - N_{f,i} \,, \tag{2.6}$$

where $N_{c,i}$ and $N_{f,i}$ are the number of colors and flavors, respectively, on the i-th node.

Seiberg duality gives rise to equivalent theories and in particular preserves the property of conformal invariance. To compute the anomalous dimensions one can, for example, use the Leigh-Strassler procedure.[2] Denote the anomalous dimension of the i-th fields by γ_i. These are related to the scaling dimension by $D_i = 1 + \frac{1}{2}\gamma_i$. Using the fact that the superpotential is always cubic we have $\gamma_1 + \gamma_2 + \gamma_3 = 0$. The numerator of the NSVZ beta function then reads

$$\beta_i = 3n_i - 3n_{i-1}n_{i+1} + \frac{3}{2} n_{i-1}n_{i+1}(\gamma_{i-1} + \gamma_{i+1}) = 3n_i - 3D_i n_{i-1}n_{i+1} \,. \tag{2.7}$$

[2] We thank Andreas Karch for a discussion on this point.

Equating this to zero gives the expression for the scaling dimensions

$$D_i = \frac{n_i^2}{n_1 n_2 n_3}\,. \tag{2.8}$$

As a check of this result we can verify that only solutions to the Diophantine equation (2.4) satisfy $D_1 + D_2 + D_3 = 3$ as required by the existence of such terms in the superpotential.

In all the computations which follow we will consider Seiberg duality which applies only to cases in which the gauge group factors are non-abelian. For this reason we will need to have at least 2 and in general N D3 branes placed at the singularity. However the value of N does not play any crucial role in the subsequent discussion and we will set it to 1 for convenience of the computation. We should have in mind that at any step of the computation it is possible to restore N to any desired value and that any discussion about Seiberg duality which make sense apply only to the SU part of the group and to $N \geq 2$.

With this said, we will consider the gauge theory for a single D3-brane, i.e., $(n_1, n_2, n_3) = (1, 1, 1)$, with respect to the canonical basis of fractional branes. Applying Seiberg duality on various nodes leads to other descriptions with different gauge groups and matter content. All these theories are presumably conformal, as explained above. In order to obtain a relation between these various Seiberg dual theories, we introduce a fictitious scale μ, and let the couplings of the gauge theory flow with μ according to the naive beta functions (2.6), applying Seiberg duality whenever one of the gauge couplings diverges. In this way, we obtain a well-defined *duality cascade*, that ends in the 'IR', $\mu \leq \mu_0$, at the canonical description (N, N, N). Going back towards the 'UV', our cascade then depends on the initial conditions at the scale μ_0, i.e., the gauge couplings, one of which must diverge. We imagine μ_0 being much smaller than the Planck scale.

We note that in the physical theory, the expected behavior is that for any given initial conditions in any Seiberg dual quiver description of the (N, N, N) quiver, the theory will flow to the conformal fixed point that is Seiberg dual to the original one. While this is difficult to show in practice, the above arguments make it plausible.

3. Duality cascades

3.1 Seiberg duality transformations

Consider applying Seiberg duality to the gauge group on the i-th node of the quiver. The number of colors and flavors for the i-th node are given by $N_{c,i} = n_i$ and $N_{f,i} = n_{i+1} f_{i-1} = n_{i-1} f_{i+1}$. Seiberg duality maps $N_{c,i} \mapsto N_{f,i} - N_{c,i}$. Using the anomaly cancellation condition (2.5), this translates into

$$(n_{i-1}, n_i, n_{i+1}) \mapsto (n'_{i-1}, n'_i, n'_{i+1}) = (n_{i-1}, 3n_{i+1}n_{i-1} - n_i, n_{i+1}) \tag{3.1}$$

$$(f'_{i-1}, f'_i, f'_{i+1}) = 3(n'_{i-1}, n'_i, n'_{i+1}) \tag{3.2}$$

Equation (3.1) is really the simplest way of writing the elementary duality transformation. However, to understand in general what is going on at the level of D-branes, their charges, and the relation to closed strings, it is necessary to keep in mind that there are various other more powerful descriptions of this duality. These formulations include Picard-Lefshetz monodromy, toric duality, Weyl reflections, tilting equivalence of derived categories, etc.. We will not go into full details here, but just mention that the precise relation between these various duality transformations is rather intricate and apparently not fully understood at present.

3.2 Coupling constants and energy scale

We define $t = \ln \mu$ and parameterize the gauge coupling of the i-th gauge group by

$$x_i = \frac{1}{g_i^2}\,. \tag{3.3}$$

We then have the RG flow equation (2.6)

$$\dot{x}_i = \frac{dx_i}{dt} = \beta_i = 3N_{c,i} - N_{f,i}\,, \tag{3.4}$$

where $N_{c,i}$ and $N_{f,i}$ are the number of colors and flavors on the i-th node, respectively,

$$N_{c,i} = n_i \tag{3.5}$$

$$N_{f,i} = n_{i+1}f_{i-1} = n_{i-1}f_{i+1} = 3n_{i+1}n_{i-1}\,. \tag{3.6}$$

Thus,

$$\dot{x}_i = 3n_i - 3n_{i+1}n_{i-1}\,. \tag{3.7}$$

It is instructive to check the beta function for the string coupling g_s. In the canonical description of the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold ($n_i = 1$) the relation between the inverse gauge couplings and the string coupling is given by

$$\frac{1}{g_s} = \sum_{i=1}^{3} \frac{1}{g_i^2}\,. \tag{3.8}$$

This formula generalizes to the quiver of figure 1 as

$$x \equiv \frac{1}{g_s} = \sum_{i=1}^{3} x_i n_i\,. \tag{3.9}$$

From this and equation (3.7) we can compute the corresponding beta function for the string coupling

$$\dot{x} = 3\left(\sum_{i=1}^{3} n_i^2 - 3n_1n_2n_3\right) = 0\,. \tag{3.10}$$

The expression in brackets vanishes since it is precisely the Diophantine equation (2.4) for the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold as explained in detail in [11]. The result is that the string coupling stays constant along the flow.

3.3 The cascade

As explained above, we start our (inverse) duality cascade at $(n_i) = (1,1,1)$, $(f_i) = (3,3,3)$ by specifying three gauge couplings, i.e., $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$, and applying Seiberg duality to one of the nodes (without loss of generality, let us say the third). We note immediately that this actually requires $x_3^{(0)} = 0$ at the scale μ_0, so that we have one less initial condition. At the first step, we then get $(n_i) = (1,1,2)$, $(f_i) = (3,3,6)$ (henceforth, we shall suppress the f_i). The theory starts flowing according to (3.7) with $t = \ln\mu$. We have

$$\dot{x} = (\beta_1, \beta_2, \beta_3) = (-3, -3, 3) \tag{3.11}$$

so that two couplings grow and one decreases. The next step in the cascade happens when one of the two inverse couplings x_1, x_2 reaches zero. Which node we dualize on will depend in the initial conditions. Let us assume that $x_1^{(0)} > x_2^{(0)}$. Then the second dualization will happen after $\Delta t = x_2^{(0)}/3$, a point at which $x_1 = x_1^{(0)} - x_2^{(0)}$, $x_3 = x_2^{(0)}$, and $n = (1,5,2)$, and so on.

Before we proceed, we show that always two of the couplings grow towards the UV, and one decreases [5]. Consider an arbitrary step in the cascade, say we dualize on the second node at $t = t_*$. For $t < t_*$, $\beta_2/3 = n_2 - n_1 n_3 < 0$, and we assume that one of $\beta_{1,3}$ is positive, and one negative. After the duality, $t > t_*$, we have

$$\frac{\beta_1'}{3} = n_1' - n_2' n_3' = n_3(n_2 - n_1 n_3) + n_1(1 - 2n_3^2) < 0 \tag{3.12}$$

$$\frac{\beta_2'}{3} = n_2' - n_1' n_3' = 2n_1 n_3 - n_2 > 0 \tag{3.13}$$

$$\frac{\beta_3'}{3} = n_3' - n_1' n_2' = n_1(n_2 - n_1 n_3) + n_3(1 - 2n_1^2) < 0\,. \tag{3.14}$$

By induction, we see that at every stretch of the cascade, two couplings grow towards the UV, and one decreases.

Our cascade is essentially a dynamical system given by (3.7), with the prescription to apply the duality (3.1) whenever any of $x_i = 0$. One may view this dynamical system as a "billiard", in which $x_i = 0$ behave like walls, with nonelastic reflections at the walls (but note that also the position of the walls of the billiard change after each reflection). The question we would like to answer is how the n_i behave as a function of "time" t, for given initial conditions $x_i^{(0)}$. One might suspect this dependence to be rather sensitive, and the billiard to display "fractal" behavior. We will see that this need not be and that there are some quantities with rather simple behavior.

We make contact with [5] by noting that one can represent all possible cascades by a "tree", in which at each node one ingoing branch splits into two outgoing branches, encoding the sequence of nodes of the quiver that one dualizes on. Our point of view here is that this tree can be organized in a very efficient and physically well motivated way by using the naive beta functions. More precisely, the initial conditions select a specific branch of the tree, making the system deterministic.

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4. One branch of the tree

Instead of being general, we now analyze one particular branch of the above mentioned tree, in which only nodes 2 and 3 participate. We will see below that this can only be achieved by choosing the singular initial condition $1/x_1^{(0)} = 0$. To this end, we will first give a slight reformulation of the dualities that involves moving the nodes. Then we solve for this branch of the tree explicitly.

4.1 The cascade in terms of (p,q) charges

One systematic way of describing the duality cascade is by using the method of (p,q) webs which was introduced in [10] and was discussed in detail in [11, 12]. Let us review the essential details needed for the present discussion.

Given a set of charges, $(p_i, q_i), i = 1, \ldots, 3$, define the intersection matrix I_{ij} by

$$I_{ij} = p_i q_j - p_j q_i \,. \tag{4.1}$$

This intersection matrix is an antisymmetric matrix with entries that encode the quiver data,

$$f_i = \epsilon_{ijk} I_{jk} \,. \tag{4.2}$$

This equation implies that the set of ranks of the gauge groups, n_i, is a null vector of the intersection matrix I_{ij} as required by the anomaly cancellation condition (2.5),

$$I_{ij} n_j = 0 \,. \tag{4.3}$$

In terms of (p,q) charges, this anomaly cancellation condition gets the form

$$p_i n_i = 0 \,, \tag{4.4}$$

$$q_i n_i = 0 \,. \tag{4.5}$$

To make connection with the RR charges e discussed in section 2.1 we note that setting [10]

$$e_1 = \frac{(p-q)^2 - 9(q^2-1)}{18q} \,, \tag{4.6}$$

$$e_2 = \frac{p-q}{3} \,, \tag{4.7}$$

$$e_3 = q \,, \tag{4.8}$$

provides a consistent set of the RR charges as expressed in section 2.1.

Seiberg duality on a single node in the quiver is a Picard-Lefshetz monodromy on the (p,q) charges. Suppose we start by dualizing node 3. This is done by a monodromy action of the (p_2, q_2) charges on the charges (p_3, q_3). Define d to be

$$d = p_2 q_3 - p_3 q_2 \,. \tag{4.9}$$

(We really have $d = 3$, but the discussion for general d is the same.) Then the matrix B in equation (2.1) is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & \mathrm{d} \end{pmatrix} . \tag{4.10}$$

The new charges (p', q') are given by the matrix action

$$p'_i = B_{ij} p_j \,, \tag{4.11}$$
$$q'_i = B_{ij} q_j \,. \tag{4.12}$$

The first row just keeps the charges (p_1, q_1) as spectators, not involved in the monodromy action. The second row replaces position of the 3rd set of charges with the second. An additional minus sign comes from the fact that the ranks, or node numbers, n_i, change sign in order to satisfy the anomaly cancellation condition, (4.4). The third row corresponds to the Picard-Lefshetz monodromy. Note that the 2nd and 3rd gauge groups have replaced their label. It is reassuring to find that using these notations the new intersection matrix I' is given in equation (2.3), and the new node numbers are given in equation (2.2). Furthermore, the new node numbers n' and new charges (p', q') satisfy the anomaly cancellation conditions (4.4). It is easy to verify that the 23 matrix entry is not changed, $I_{23} = I'_{23}$.

4.2 Explicit solution

The next step in the cascade that we are describing is given by another application of the matrix B, (4.10) on the charges (p', q'), and the full duality cascade simply becomes the successive application of the matrix B. In order to solve explicitly this 'one branch of the tree' let us denote the node numbers n_i after the k-th step of the duality cascade by $n_i^{(k)}$. They are simply obtained from the initial node numbers $n_i^{(0)} = 1$ by the application of the k-th power of the matrix B to the initial charges (p, q), i.e.,

$$n_i^{(k)} = n_j^{(0)} B_{ji}^{-k} \,. \tag{4.13}$$

By diagonalizing B, one can get an expression for these numbers in terms of the eigenvalues λ of the matrix B. These eigenvalues are the solutions of the quadratic equation

$$\lambda^2 - d\lambda + 1 = 0 \,, \qquad \lambda_\pm = \frac{1}{2}(d \pm \sqrt{d^2 - 4}) \,. \tag{4.14}$$

We then obtain for the node numbers

$$n_1^{(k)} = 1 \,, \tag{4.15}$$
$$n_2^{(k)} = n_3^{(k+1)} \,, \tag{4.16}$$
$$n_3^{(k)} = \frac{\lambda_+^k - \lambda_-^k + \lambda_-^{k-1} - \lambda_+^{k-1}}{\lambda_+ - \lambda_-} \,. \tag{4.17}$$

Before we proceed it is interesting to observe the behavior of the cascade as a function of d. Clearly, $d = 2$ is a critical number which gives eigenvalues 1 and therefore produces a linear growth of $n^{(k)}$ as a function of k. For $d \geq 3$ there is an exponential growth while for $d = 1$ the eigenvalues are complex and we expect some critical change in behavior. These phenomena do not happen for the present case of study which is for $\mathbb{C}_3/\mathbb{Z}_3$. In more complicated geometries we would expect to find some of this interesting pattern to appear. In the terminology of [6] which is mentioned in the introduction, the cases $d = 2, d > 2, d < 2$ correspond to an elliptic, hyperbolic and parabolic Cartan matrix, respectively. Furthermore, the duality cascade studied in [3] has $d = 2$ and therefore is elliptic, corresponding to a linear growth of the rank in the number of duality steps.

In order to trace the (unphysical) energy scale μ along the cascade, we need to compute the naive beta functions at each step. We denote the beta function of the i-th group between the k-th and k+1-st step of the duality cascade, as in equation (3.4), by $\beta_i^{(k)}$, and similarly, the inverse gauge couplings of equation (3.3) by $x_i^{(k)} = 1/(g_i^{(k)})^2$. The beta functions after the k-th step of the cascade are then

$$\beta_1^{(k)} = 3\big(1 - n_2^{(k)} n_3^{(k)}\big) = -3\big(n_2^{(k)} - n_3^{(k)}\big)^2 < 0\,, \tag{4.18}$$

$$\beta_2^{(k)} = 3\big(n_2^{(k)} - n_3^{(k)}\big) > 0\,, \tag{4.19}$$

$$\beta_3^{(k)} = 3\big(n_3^{(k)} - n_2^{(k)}\big) = -\beta_2^{(k)} < 0\,. \tag{4.20}$$

The second equality in the first line is made using the Diophantine equation in (3.10). It is easy to see that β_1 and β_3 are negative while β_2 is positive, independent of k. To write down the solution of the equations $\dot{x}_i^{(k)} = \beta_i^{(k)}$, we define the energy scale at which the k-th step of the duality cascade is performed by t_k. The initial conditions for these equations are the inverse gauge couplings $x_1^{(1)}(t_1) = x_1^{(0)}$, $x_2^{(1)}(t_1) = x_3^{(0)}$, and $x_3^{(1)}(t_1) = x_2^{(0)}$, where $t_1 = \ln \mu_0$ is the (arbitrary) scale at which we start the cascade. Taking into account the permutation of the 2nd and 3rd gauge groups at each step of the cascade, we find for $t_k < t < t_{k+1}$

$$x_1^{(k)}(t) = \beta_1^{(k)}(t - t_k) + x_1^{(k-1)}(t_k)\,, \tag{4.21}$$

$$x_2^{(k)}(t) = \beta_2^{(k)}(t - t_k) + x_3^{(k-1)}(t_k)\,, \tag{4.22}$$

$$x_3^{(k)}(t) = \beta_3^{(k)}(t - t_k) + x_2^{(k-1)}(t_k)\,. \tag{4.23}$$

The k+1-st step of the cascade is done when one gauge coupling diverges. The signs of the beta function then imply that it is always the third gauge group which is dualized. The condition becomes $x_3^{(k)}(t_{k+1}) = 0, k \geq 1$. Taking this into account and combining the second and third equation after setting $t = t_k$ we find that, using (4.20),

$$\beta_2^{(k)}(t_{k+1} - t_k) = \beta_2^{(k-1)}(t_k - t_{k-1}) = \cdots = \beta_2^{(2)}(t_3 - t_2) \tag{4.24}$$

$$= \beta_2^{(1)}(t_2 - t_1) + x_3^{(0)} = x_2^{(0)} + x_3^{(0)}. \tag{4.25}$$

This difference equation then solves for the energy scale at the k+1-st step

$$t_{k+1} = t_1 - \frac{x_3^{(0)}}{\beta_2^{(1)}} + \big(x_2^{(0)} + x_3^{(0)}\big) \sum_{j=1}^{k} \frac{1}{\beta_2^{(j)}}\,. \tag{4.26}$$

This result can be rewritten using the explicit expressions for the beta functions in terms of the eigenvalues of the monodromy matrix. We have

$$\beta_2^{(k)} = 3\frac{\lambda_+^{k+1} - \lambda_-^{k+1} + 2\lambda_-^k - 2\lambda_+^k - \lambda_-^{k-1} + \lambda_+^{k-1}}{\lambda_+ - \lambda_-} = 3(d-2)\frac{\lambda_+^k - \lambda_-^k}{\lambda_+ - \lambda_-}\,, \tag{4.27}$$

and hence

$$t_{k+1} = t_1 - \frac{x_3^{(0)}}{3} + \left(x_2^{(0)} + x_3^{(0)}\right)\frac{(\lambda_+ - \lambda_-)}{3(d-2)}\sum_{j=1}^{k}\frac{1}{\lambda_+^j - \lambda_-^j}\,. \tag{4.28}$$

Let us study the values of the inverse gauge couplings at the energy scales t_k. According to our boundary condition, $x_3^{(k)}(t_{k+1}) = 0$. Using equations (4.22) and (4.24) we find that $x_2^{(k)}(t_{k+1}) = x_2^{(0)} + x_3^{(0)}$, independent of k. The most interesting result is for $x_1^{(k)}(t_{k+1})$. We can either compute it directly using equations (4.21) and (4.26), or by observing that the weighted sum of the couplings, equation (3.9), is constant along the flow. This value can simply be computed for $k = 1$ and we summarize the results below.

$$x_1^{(k)}(t_{k+1}) = x_1^{(0)} - (n_2^{(k)} - 1)x_2^{(0)} - (n_2^{(k)} - 2)x_3^{(0)}\,, \tag{4.29}$$

$$x_2^{(k)}(t_{k+1}) = x_2^{(0)} + x_3^{(0)}\,, \tag{4.30}$$

$$x_3^{(k)}(t_{k+1}) = 0\,, \tag{4.31}$$

$$\frac{1}{g_s} = \sum_{i=1}^{3} n_i^{(k)} x_i^{(k)} = x_1^{(0)} + 2x_3^{(0)} + x_2^{(0)}\,. \tag{4.32}$$

Since $n_2^{(k)}$ grows without bounds, this computation demonstrates that the inverse coupling of the first gauge group, the one which is not participating in the cascade along the specific branch that we have been considering, reaches zero at some point, and does so in an exponential fashion. Therefore, this cascade can not go on forever without involving the first gauge group—unless we had actually made $x_1^{(0)} = \infty$. In this case, which is a special case of the ones considered in [6], we easily see from (4.28) and the fact that $\lambda_+ > 1$ for $d = 3$, that $\lim_{k\to\infty} t_k < \infty$. This is the simplest illustration of the wall phenomenon.

5. The Wall

The conclusion of the previous section forces us to consider cascades that involve all three nodes. Unfortunately, the combinatorics become quite involved, and we have not been able to solve explicitly for the general cascade. But our systems lends itself naturally to a numerical study, since, as it turns out, the dualities converge exponentially towards the wall.

5.1 The position of the wall

The (numerical) determination of the position of the wall becomes extremely simple once we remove the redundancy from the initial conditions. Recall that we were already forced to make $x_3^{(0)} = 0$. Moreover, it is easy to see from the homogeneity of the equations (3.4), and from the explicit computation in the previous section that we may rescale $x_2^{(0)}$ to one.

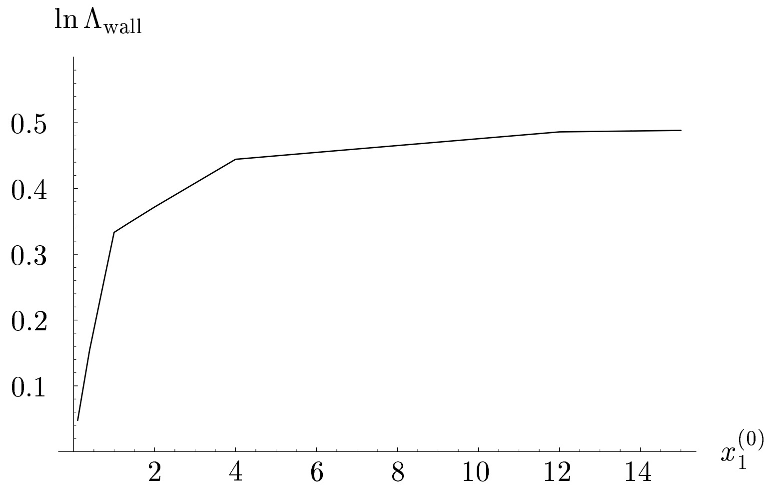

Figure 2: The position of the duality wall as a function of initial condition $x_1^{(0)}$. The breaking points are located at $x_1^{(0)} = 1, 4, 12, 33, \ldots$, and can be calculated from equation (4.29) as $n_2^{(k)} - 1$. The asymptotic value $\lim_{x_1^{(0)} \to \infty} t_{\text{wall}}$ can be computed from (4.28) to be $.5117\ldots$.

From the point of view of the dynamical system, this simply amounts to a rescaling of "time" t. From a physical point of view, this rescaling is not the choice of an energy scale, which is $t_1 = \ln \mu_0$, but rather amounts to a rescaling of the string coupling g_s. In any case, this leaves us with a single initial condition $x_1^{(0)}$. Finally, we note that we may also restrict ourselves to $x_1^{(0)} > 1$, since otherwise we simply exchange $x_1^{(0)}$ and $x_2^{(0)}$.

With these initial conditions, the cascade proceeds as follows. As long as $x_1 > 0$, we are on the 'branch of the tree' described in the previous section. When x_1 reaches zero between the k-th and k+1-st step of the cascade, we have to start including the first node in the cascade, and we do not know the general solution.

Our findings for the position of the wall $t_{\text{wall}} = \ln \Lambda_{\text{wall}}$ as a function of the initial condition are shown in figure 2. Surprisingly, the function $t_{\text{wall}}(x_1^{(0)})$ is simply piecewise linear. The points of discontinuity can be traced back to the explicit solution of the 'branch of the tree' in the previous section. More precisely, it appears that the cascade becomes singular for those initial conditions for which the first node starts to play a role, i.e., x_1 reaches 0, at the same time t_{k+1} at which we would have had to perform the next step of dualizing node 3. In other words, the breaking points in figure 2 can be found by setting $x_1^{(k)}(t_{k+1})$ to zero in (4.29) and solving for $x_1^{(0)}$. One then finds, using $x_2^{(0)} = 1$ and $x_3^{(0)} = 0$, that the special initial conditions are given by

$$x_1^{(0),k} = n_2^{(k)} - 1\,, \tag{5.1}$$

where $n_2^{(k)}$ is given by equations (4.16) and (4.17). For $d = 3$, this becomes the sequence $1, 4, 12, 33\ldots$. (in figure 2, there are also breaking points at the inverses of these numbers, because of the symmetry $1 \leftrightarrow 2$). Moreover, we note that the position of the wall for these special initial conditions $x_1^{(0),k}$ can also be determined from the results of the previous section and are given by equation (4.28).

At present, we do not understand the precise mechanism that leads to discontinuities in $t_{\rm wall}(x_1^{(0)})$ at these special points, nor the linear behavior between them. In fact, given the origin of the $x_1^{(0),k}$ mentioned above, it would have been natural to suspect that there be further breaking points whenever two inverse couplings reach zero at the same time, possibly after a fairly complicated sequence of dualities involving all three nodes. We have not been able to detect such a 'fractal' behavior.

5.2 Critical exponent

Given that we have found a simple description for the position of the duality wall as a function of the initial condition $x_1^{(0)}$, it is a natural question to ask how the wall is approached, i.e., how do the node numbers n_i's diverge as $t \to t_{\rm wall}$. Most naively, one expects a power law behavior

$$n_i(t) \sim \frac{1}{(t_{\rm wall} - t)^{\gamma_i}}\,, \tag{5.2}$$

and one can ask how the γ_i depend on the initial conditions. From a physical point of view, the 'critical exponents' γ_i measure the growth in the number of degrees of freedom during the approach of the wall. We can then also reduce (5.2) to a single number, and study

$$n_1(t)\, n_2(t)\, n_3(t) \sim \frac{1}{(t_{\rm wall} - t)^{\gamma}}\,, \tag{5.3}$$

with $\gamma = \sum_{i=1}^{3} \gamma_i$. It is clear that (5.3) is not the only possible definition of γ, and that one could imagine other measures of the 'number of degrees of freedom'. In a thermodynamical approach, however, the precise definition should not matter too much.

As an example, let us look at the vicinity of the breaking points in $t_{\rm wall}(x_1^{(0)})$. It is easy to see that, say for $k = 2$, the sequence of nodes that we dualize on is given by

$$(321313131\ldots) \qquad \text{for } x_1^{(0)} \lesssim x_1^{(0),2} \tag{5.4}$$

$$(323131313\ldots) \qquad \text{for } x_1^{(0)} \gtrsim x_1^{(0),2}\,. \tag{5.5}$$

From this, one might be tempted to conclude that the second node is not dualized on close to the wall, and n_2 is constant, which would imply that γ_2 goes to zero at $x_1^{(0),2}$. However, as we have seen in the previous section, as long as the initial conditions are not singular (as for example at $x_1^{(0)} = x_1^{(0),2}$), the cascade cannot proceed only with nodes 1 and 3. The second node must eventually participate again.

In fact, it seems that rather than depending on initial conditions, the critical exponents are actually constant. More precisely, the numerics indicate that

$$-\frac{\ln n_1^{(k)} n_2^{(k)} n_3^{(k)}}{\ln(t_{\rm wall} - t_k)} \longrightarrow 1 \qquad \text{as } k \to \infty\,, \tag{5.6}$$

where k numbers the steps in the general cascade starting at initial condition $x_1^{(0)}$ and ending at $t_{\rm wall}$. We do not have an analytical proof of (5.6), but the following heuristic arguments show that it is a sensible result.

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We have seen that in the generic cascade, we must always include all three nodes, and therefore all three node numbers should grow in roughly equal proportions. It makes sense, therefore, to consider appropriately averaged quantities,[3] which we will denote by dropping the subscript i. For example, in view of (5.3), one could consider the geometric average, i.e., $n^{(k)} = (n_1^{(k)} n_2^{(k)} n_3^{(k)})^{1/3}$, etc.. Let us assume that in this averaged sense, the node numbers grow exponentially with k, i.e.,

$$\frac{\Delta n^{(k)}}{\Delta k} \sim n^{(k)} \,. \tag{5.7}$$

Furthermore, we know from (3.9) that the x_i are inversely proportional to the n_i's, i.e., $x^{(k)} \sim 1/n^{(k)}$, while the beta functions grow quadratically in $n^{(k)}$. Hence,

$$\frac{\Delta t_k}{\Delta k} \sim \frac{x^{(k)}}{\beta^{(k)}} \sim \left(n^{(k)}\right)^3 . \tag{5.8}$$

Combining (5.7) and (5.8), we find

$$\frac{\Delta n(t)}{\Delta t} \sim n(t)^4 \,, \tag{5.9}$$

which indeed yields (5.6), i.e., $n(t) \sim 1/(t_{\text{wall}} - t)^{1/3}$.

We note one caveat toward the result (5.6), which comes from the asymptotics along the special 'branch of the tree' described in the previous section. Indeed, from (4.16), (4.17), and (4.28), we find that for $k \to \infty$, $t_\infty - t_k \sim 1/\lambda_+^k$, while $n_1^{(k)} = 1$, and $n_2^{(k)}, n_3^{(k)} \sim \lambda_+^k$. This would imply $\gamma = 2$. We attribute the discrepancy to the fact that $x_1^{(0)} \to \infty$ is a singular initial condition.

6. Conclusions and open questions

In this paper, we have studied some simple properties of the dynamical system (3.7) associated by naive RG flow with the quiver describing D3-branes at the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold singularity. The system exhibits the phenomenon of duality walls. The number of degrees of freedom grows exponentially as a function of the number of steps in the cascade. On the other hand, the growth in scale decreases exponentially at the same rate as the number of degrees of freedom. This results in the piling up of dualities at a "duality wall", i.e., the number of degrees of freedom grows faster than exponential as a function of energy scale.

As mentioned in the introduction, our results do not have any direct implications concerning the existence of duality walls in string theory. These duality walls, introduced in [7] and studied in more detail in [6], thus await their realization in string theory. While one could imagine that D3-branes at a generic threefold singularity with a hyperbolic quiver would be the appropriate framework for this, our present results are insufficient. Nevertheless, our results give an illustration of the phenomenon, and at the very least our flows can be viewed as an organizing principle for the tree of Seiberg dualities.

[3] In the general context of billiards and similar dynamical systems, one has to be extremely careful with averagings of this sort. Here, they seem to give sensible results.

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We have in particular studied the dependence of the position of the wall on the initial conditions of the cascade, and have found that after the appropriate rescalings, the dependence is simply piecewise linear. Moreover, the approach to the wall is found to be governed by a simple scaling behavior (5.6). While these results are intriguing, in absence of a deeper understanding of the duality walls, it is hard to give a physical interpretation for them.

We can imagine several approaches to the goal of answering some of these physical questions. For instance, it would be useful to understand the generic cascade analytically, generalizing our results for the 'branch of the tree' involving only two nodes. In particular, one could try to verify more rigorously the results on the position of the wall and the approach to it. In the quest for further structure, and for a possible physical realization of the walls, it will be helpful to repeat this analysis for other quivers from branes at singularities, such as those arising from contracting del Pezzos in Calabi-Yau threefolds.

More input can also be expected from looking for possible holographic duals of our theory. Admittedly, since the theory on regular D3-branes at the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold singularity is conformal, the holographic dual is simply $AdS_5 \times S^5/\mathbb{Z}_3$. Our cascades might be related to duals of irrelevant deformations of such a background, and should make predictions, for instance, about black hole entropy in this context.

Finally, we note that it would be interesting to look for the special points in the Kähler moduli space corresponding to the duality walls. Such points will be found because $x_1^{(0)}$ is the (appropriately rescaled) gauge coupling, hence related to the B-field. What does the position of the wall $t_{\rm wall}(x_1^{(0)})$ mean in Kähler moduli space?

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